COMPACTNESS AND EFFECTIVITY FOR FUZZY LOGICS: DISCUSSING ON SOME CRITICISMS

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Abstract: I consider some questions suggested by some criticisms against fuzzy logic (in narrow sense). Mainly, I discuss about the appropriate notions of compactness and effectiveness for fuzzy logic.

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1. Introduction

The name "fuzzy logic in narrow sense", in brief "fuzzy logic", denotes several different proposals of formal logical systems to handle vague predicates in a rigorous way. Now, in spite of a large series of interesting results, there are again several criticisms against this topic. The main part of these criticisms arise from misconceptions, in my opinion. Nevertheless there are criticisms suggesting the necessity for a new adequate basis for fuzzy logic.

As an example of a misconception we quote F. J. Pelletier (see [26, 27]) who, in answering a paper of C. W. Entmann [3] and in reviewing the basic book of P. Hájek [14], disputes Entmann's claim that fuzzy logic is an extension of the classical one:

"Most logicians think that one logic is an extension of another if it contains all the theorems of the other ... But this is not a sense in which fuzzy logic is an extension of classical logic; for, ($A\vee\neg A$) *is a theorem of classical logic but not of fuzzy logic. Indeed, it can be shown that there is no theorem of fuzzy logic* (*in* $\wedge$, $\vee$, $\neg$) *which is not already a theorem of classical logic. So classical logic in fact is an extension of fuzzy logic, in the usual use of the term 'extension', and not the other way around."*.

On the other hand the conviction that fuzzy logic is a *deviant* logic and not only an attempt to extend classical logic is one of the reasons of the negative judgment of many philosophers. As an example, in speaking about the solution of the Heap paradox proposed by fuzzy logic (see Goguen [9] and Hájek and Novák [16]), R. Sorensen in [29] claims

"I am a logical conservative in that I deny that vagueness provides any reason to reject any theorem or inference rule of standard logic".

In accordance *"instead of changing logic, we should change our opinions about how language works"*. Notice that the word used is *"changing"* and not *"enlarging"*. Instead, in my opinion, it is correct to consider fuzzy logic as an attempt to extend the sphere of action of classic logic to admit predicate which are "vague" (more precisely *graded*) in nature. In this sense it is not at all a deviant logic. The basic question is the adequateness of such an attempt and not a possible conflict with classical logic.

More relevant criticisms are related with compactness and effectiveness: two basic notions in any logic. Indeed, Pelletier observes that while classical logic is semantically compact this is not the case of fuzzy logic

"where there can be (*for example*) *an unsatisfiable infinite set* Γ *where every finite set is satisfiable"*.

On the other hand,

"since all proofs are by definition finite, there can therefore be no general proof theory for fuzzy logic".

Also, while the set of valid formulas in classical logic is recursively enumerable:

"Hájek then shows that no similar result is possible for either the Łukasiewicz or Goguen logics: no recursive axiomatization for either of these logics is possible …

Pelletier emphasizes the importance of this fact for fuzzy logic:

"Many observers might think this should the death knell for fuzzy logic" [26].

Notice that, in spite of these observations, the global opinion expressed by Pelletier on Hajek's book is very positive.

In this position paper we observe that in the literature there are two kinds of answers to these questions corresponding to two different approaches to fuzzy logic. The first one (and the more extensively embraced by the fuzzy logic community) is referred to Hájek's school and it characterized by deduction apparatus which are crisp in nature. In such a case to obtain the compactness and the effectiveness one passes from a fixed valuation algebra to the valuation algebras belonging to the variety generated by such an algebra (see [2, 15, 18, 19, 20]). The latter approach refers to graded deduction apparatus in accordance with the ideas of J.A. Goguen and J. Pavelka (see [9, 25] and also [7, 21, 22, 23, 24, 30]) where the valuation algebra is fixed and graded set of hypotheses and graded inference rules are admitted. In such a case I argue that there is no difficulty to obtain compactness and effectiveness provided that correct definitions are adopted. These definitions, suggested by the theory of effective domains (see [1, 7, 8]), are the correct ones since the notion of computability in the framework of a continuous set of truth values have to be totally different from the notion of computability in the framework of the discrete set {0,1}. The first one is related with the idea of an endless effective approximation process, the latter with the idea of an effective process giving the exact result after a finite number of steps.

2. Preliminaries

We recall some elementary notions on fuzzy logic (see [12, 14, 23, 32] for a detailed exposition). Let L be a bounded lattice and S a nonempty set, then we call *L-subset* or *fuzzy subset of S* any map $s : S \to L$ from S into L (see [32]). Given a natural number n, an *n-ary L-relation* or *fuzzy relation* is an L-subset of S^n. We denote by L^S the class of all the L-subsets of S and by $P(S)$ the class of all the subsets of S. Given $\lambda \in L$, the *open λ-cut* of s is the set $\{x\in S : s(x)>\lambda\}$, the *closed λ-cut* the set $\{x\in S : s(x)\geq\lambda\}$. We say that s is *finite* if $Supp(s) = \{x\in S : s(x) \neq 0\}$ is finite. Given two L-subsets s_1 and s_2, we set $s_1\subseteq s_2$ provided that $s_1(x)\leq s_2(x)$ for every $x\in S$ and in such a case we say that s_1 *is contained in* s_2. The pair $(L^S, \subseteq)$ is a bounded lattice whose meets and joins we call *intersection* and *union.* We denote by $\cap$ and $\cup$ these operations and therefore we set, for every s_1 and s_2 in L^S and $x\in S$,

$$(s_1\cup s_2)(x) = s_1(x)\vee s_2(x) \ \ ; \ \ (s_1\cap s_2)(x) = s_1(x)\wedge s_2(x).$$

If 0 and 1 are the bounds in L, we call *crisp* any L-subset whose values are in {0,1}. We associate every subset X of S with a crisp L-subset, i.e. its characteristic function $c_X\in \{0,1\}^S$ defined by setting $c_X(x) = 1$ if $x\in X$ and $c_X(x) = 0$ otherwise. In accordance, we identify the class $P(S)$ of all subsets of S with the class $\{0,1\}^S$ of crisp fuzzy subsets of S.

Two main approaches to fuzzy logic exist. In the former the deduction apparatus works on a crisp set of hypotheses to give the related crisp set of consequences (proposed by P. Hájek and others). In the latter the deduction apparatus works on a fuzzy subsets of hypotheses to give the related fuzzy subset of consequences (proposed by G. Goguen, J. Pavelka and others). We refer to them as *ungraded approach*, in brief *U-approach*, and *graded-approach*, in brief *G-approach*, respectively. We prefer such a terminology to the usual one of "with classical syntax" and "with evaluated syntax" since the difference is semantic in nature and it is a consequence of different definitions of the entailment relation. In both the cases one considers a first order language and *valuation algebras*, i.e. bounded lattices equipped with suitable operations to interpret the logical connectives. In this paper we refer to fuzzy logics whose language coincides with the classical one extended with possible logical constants. Also we consider the most important class of valuation algebras, the *standard algebras*, i.e. algebras whose domain is the real interval [0,1] and whose operations are given by a continuous t-norm $\otimes$ together with the related residuum $\to$. A basic example is the *Łukasiewicz's product* $\otimes$ where $\lambda\otimes\mu = (\lambda+\mu-1)\vee 0$ and $\lambda\to\mu = (1-\lambda+\mu)\wedge 1$, for every $\lambda, \mu \in [0,1]$.

Given a valuation algebra and a first order language, we call *L-interpretation* or *fuzzy interpretation* a pair (D,I) where

- D is a set we call *the domain* of the interpretation
- $I(r)$ is an n-ary L-relation in D for any n-ary relation name r
- $I(c)$ is an element in D for any constant c
- $I(f)$ in an n-ary function in D for any n-ary function name f.

Usually one assumes that in the language there are logical constants to interpret suitable truth values. For example, in rational Pavelka logic all the elements in the set $[0,1]_Q$ of rational numbers in [0,1] are represented by logical constants. Denote by *Form* the set of formulas, then, given a fuzzy interpretation (D,I), we can evaluate in L the formulas in a truth-functional way, as usual. Unfortunately, in the case L is not complete it is possible that some quantified formula cannot be evaluated. This since the universal and existential quantifiers are interpreted by the greatest upper bound and by the least upper bound, respectively. We call *safe* an interpretation in which all the formulas are evaluated and in such a case we call *truth-functional valuation* the valuation $m_I : F \to L$ of the formulas induced by (D,I).

In the U-approach, given $\alpha\in$ *Form*, we say that an interpretation (D,I) is a *model of* α provided that $m_I(\alpha) = 1$. Given a set T of formulas, we say that (D,I) is *a model of T* if (D,I) is a model of any formula in T. Every standard algebra enables us to define two entailment relations.

Definition 2.1. Given a standard algebra ([0,1], $\otimes$, $\rightarrow$) denote by *Varl*($\otimes$) the class of all linearly ordered algebras in the variety generated by ([0,1], $\otimes$, $\rightarrow$). Then we call $\otimes$-*model* an interpretation in ([0,1], $\otimes$, $\rightarrow$) and *Varl*($\otimes$)-*model* an interpretation in a valuation algebra in *Varl*($\otimes$). Given $T \subseteq Form$ and $\alpha \in Form$, we write

- $T \models_{\otimes} \alpha$ provided that every safe $\otimes$-model of T is a model of α,
- $T \models_{Varl(\otimes)} \alpha$ provided that every safe *Varl*($\otimes$)-model of T is a model of α.

These two entailment relations are associated with two different notions of tautology. We write $\models_{\otimes}\alpha$ and $\models_{Varl(\otimes)}\alpha$ instead of $\emptyset \models_{\otimes}\alpha$ and $\emptyset \models_{Varl(\otimes)}\alpha$, respectively.

Definition 2.2. A formula α such that $\models_{\otimes} \alpha$ is named a *standard* $\otimes$*–tautology*. A formula α such that $\models_{Varl(\otimes)}\alpha$ is named a *general* $\otimes$*-tautology*.

Then a standard $\otimes$-tautology is a formula satisfied in all the $\otimes$-models and a general $\otimes$-tautology is a formula satisfied in all the safe *Varl*($\otimes$)-models. In first order fuzzy logic the general $\otimes$-tautologies form a subset of the set of standard $\otimes$-tautologies. The deduction apparatus in *U*-approach is defined by adopting the same paradigm of classical logic, i.e. by a set of logical axioms and suitable inference rules. This apparatus enables us to generate, given a (crisp) set of proper axioms, the related (crisp) set of theorems.

Instead in the *G*-approach, the semantics is defined as follows.

Definition 2.3. Given a standard algebra, let $\tau \in [0,1]^{Form}$ be a *fuzzy theory*, i.e. a fuzzy subset of formulas. Then we say that (D,I) *is a model of* τ, in brief $(D,I) \models \tau$, provided that $m_I \supseteq \tau$. Moreover, the *logical consequence operator* is the map $L_f : [0,1]^{Form} \rightarrow [0,1]^{Form}$ such that,

$$L_f(\tau)(\alpha) = Inf\{m_I(\alpha) : (D,I) \models \tau\}, \tag{2.1}$$

for every $\tau \in L^{Form}$ and $\alpha \in Form$. $Tau = L_f(\emptyset)$ is called the *fuzzy subset of tautologies*.

The number $L_f(\tau)(\alpha)$ is sometime called *truth degree of* α in τ. This is not a correct expression, in my opinion and this since such a number is not a truth degree but a constraint (i.e. a piece of information) on a truth degree. Indeed, observe that the aim of any logic is elaborate (incomplete) information. In the case of fuzzy logic it is reasonable that such an information is expressed by claims *as "the truth value of* α *is between* λ *and* μ" i.e. by assigning, for every formula α, an interval constraint $[\lambda, \mu]$ on the actual truth value of α. On the other hand, in the fuzzy logics with a "good" negation we can split such a constraint into the two lower-bound constraints "the truth values of α is greater or equal to λ" and "the truth values of $\neg\alpha$ is greater or equal to $\sim\mu$". Then it is not restrictive to assume that the available information is a set of lower-bound constraints on the actual truth degree of the formulas. Also, we can reduce this set to a function τ. In accordance $L_f(\tau)(\alpha)$ is the best possible lower-bound constraint on the truth degrees of the formula α given τ and it gives in an explicit way the whole information carried on by τ. Observe that the relation $\models_{\otimes}$ is in accordance with Definition 2.3 since $T \models_{\otimes} \alpha$ if and only if $L_f(T)(\alpha) = 1$. The next basic step of the *G*-approach is to define the deduction apparatus.

Definition 2.4. A *fuzzy Hilbert system* is a pair $(la, \mathcal{R})$ where la is a fuzzy subset of formulas, the *fuzzy subset of logical axioms*, and $\mathcal{R}$ is a set of fuzzy inference rules. In turn, *a fuzzy inference rule* is a pair $r = (r', r'')$, where

- r' is a partial n-ary operation on *Form*,
- r'' is an n-ary operation on [0,1] preserving the least upper bounds (continuity hypothesis).

We indicate an application of an inference rule r by the picture

$$\frac{\alpha_1,\ldots,\alpha_n}{r'(\alpha_1,\ldots,\alpha_n)} \quad ; \quad \frac{\lambda_1,\ldots,\lambda_n}{r''(\lambda_1,\ldots,\lambda_n)}$$

whose meaning is that:

IF you know that $\alpha_1,\ldots,\alpha_n$ are true (at least) to the degree $\lambda_1,\ldots,\lambda_n$
THEN $r'(\alpha_1,\ldots,\alpha_n)$ is true (at least) at level $r''(\lambda_1,\ldots,\lambda_n)$.

A *proof* π *of a formula* α is a sequence $\alpha_1,...,\alpha_m$ of formulas such that $\alpha_m = \alpha$, together with the related *"justifications"*. This means that, for any formula α_i, we must specify whether

(i) α_i is assumed as a logical axiom; or

(ii) α_i is assumed as an hypothesis; or

(iii) α_i is obtained by the first component of a rule (in such a case we have to specify the rule and the formulas in the list $\alpha_1,\dots,\alpha_{i-1}$ used by the rule).

Let τ be a fuzzy theory (the available information) and π a proof. Then the *valuation* $Val(\pi,\tau)$ *of* π *with respect to* τ is defined by induction on the length m of π as follows:

$$Val(\pi,\tau) = \begin{cases} la(\alpha_m) & \text{if } \alpha_m \text{ is assumed as a logical axiom,} \\ \tau(\alpha_m) & \text{if } \alpha_m \text{ is assumed as an hypothesis,} \\ r''(Val(\pi(i(1)),\tau),\dots,Val(\pi(i(n)),\tau)) & \text{if } \alpha_m = r'(\alpha_{i(1)},\dots,\alpha_{i(n)}) \end{cases}$$

where, $1\leq i(1)<m,\dots,1\leq i(n)<m$. If α is the formula proven by π, the meaning of $Val(\pi,\tau)$ is that:

given the information τ, π *assures that* α *holds at least at level* $Val(\pi,\tau)$.

Different proofs of the same formula α can give different valuations. Then, we have to fuse the pieces of information on the truth degree of α obtained by all the possible proofs of α.

Definition 2.5. Given a fuzzy Hilbert's system $(a,\mathcal{R})$, the operator $D_f : [0,1]^{Form} \to [0,1]^{Form}$ defined by setting

$$D_f(\tau)(\alpha) = Sup\{Val(\pi,\tau) : \pi \text{ is a proof of } \alpha\} \tag{2.2}$$

is called *the deduction operator of* $(a,\mathcal{R})$.

We are now able to give a general definition of fuzzy logic with evaluated syntax.

Definition 2.6. Given a standard algebra, we say that the related fuzzy logic *is axiomatizable* provided that there is a fuzzy Hilbert system such that $L_f = D_f$. In such a case we say also that a *completeness theorem* holds true.

There is no difficulty to extend all these definitions to any valuation algebra. Another extension is in giving an abstract definition of fuzzy semantics in accordance with Pavelka's ideas.

Definition 2.7. Given a valuation structure, a *fuzzy* semantics is a class $\mathcal{M}$ of valuations in this structure of the set of formulas.

An example is obtained by assuming that $\mathcal{M}$ is the class of truth functional valuations but there are also interesting examples of non truth-functional semantics. It is immediate how to extend all the definitions in this section by referring to any abstract fuzzy semantics.

3. Compactness

A first answer to Pelletier's claim about the non compactness and non effectiveness of fuzzy logic is simply to observe that in the case of valuation structures with finitely many elements there are several counterexamples (see for example [10]). A more complex answer is necessary in the case the set of truth degrees is infinite, for example the interval [0,1]. Indeed in the U-approach one distinguishes two notions of compactness corresponding to the two entailment relations $\models_{Varl(\otimes)}$ and $\models_{\otimes}$ (see [14]).

Definition 3.1. We say that the logic $\mathcal{L}$ associated with a standard algebra $([0,1], \otimes,\to)$ is *compact in standard sense*, provided that for every theory T and α formula,

$T \models_{\otimes} \alpha \Rightarrow$ there is a finite part T_f of T such that $T_f \models_{\otimes} \alpha$.

$\mathcal{L}$ is *compact in a general sense*, provided that

$T \models_{Varl(\otimes)} \alpha \Rightarrow$ there is a finite part T_f of T such that $T_f \models_{Varl(\otimes)} \alpha$.

The following very interesting result shows that we can avoid Pelletier's criticisms by referring to compactness in general sense (see [15]). We refer only to Łukasiewicz logic, i.e. to the case $\otimes$ is Łukasiewicz triangular norm. Nevertheless these results hold true also for other basic logics.

Theorem 3.2. *While Łukasiewicz logic is not compact in standard sense, it is compact in general sense.*

A different notion of compactness is necessary in the case we refer to the *G*-approach. Indeed in such a case the compactness is a property of the logical consequence operator $L_f : [0,1]^{Form} \to [0,1]^{Form}$ and in defining it is necessary to take in account the different topological structures of $\{0,1\}^{Form}$ and $[0,1]^{Form}$. This suggests to look at a continuity property and not at a finiteness property (see [1]). Recall the *limit* of a upward directed class *C* is defined as its least upper bound, i.e. $limC = SupC$. Also, the imagine of a upward directed class by a monotone map is an upward directed class.

Definition 3.3. Let $(L, \leq)$ be a complete lattice, then a function $H : L \to L$ is *continuous* provided that

$$H(limC) = lim\, H(C)$$

for any upward directed class *C* of elements in *L*. Given a nonempty set *S*, we call *compact* an operator $H : L^S \to L^S$ which is continuous in the lattice L^S.

Notice that this notion, which is a basic one in domain theory, it is different from the one proposed by Pavelka and it is long time known in logic programming. Indeed, it enables us to define the least Herbrand model of a program as a fixed point of the immediate consequence operator. There are several reasons to assume the continuity as the correct counterpart of the notion of compactness in fuzzy logic. Firstly, in the case of the lattice of all subsets of a given set, this notion coincides with the usual one. Moreover, we can characterize the continuity in the lattice of the fuzzy subsets of a given set in terms of finite fuzzy subsets (see [20]). Indeed, define the relation $\prec$ by setting, for any $s_1, s_2 \in L^S$,

$$s_1 \prec s_2 \iff s_1(x) < s_2(x) \text{ for every } x \in Supp(s_1).$$

Then we can prove that $H : L^S \to L^S$ is continuous if and only if, for every fuzzy subset *s*,

$$H(s) = \cup\{H(s_f) : s_f \textit{ is finite and } s_f \prec s\}. \tag{3.1}$$

Once we admit Definition 3.3, the following theorem gives an answer to Pelletier's criticism (see [1]).

Theorem 3.4. *The deduction operator of a fuzzy Hilbert system* (*in a countable language*) *is compact. Conversely, given a continuous operator H, a fuzzy Hilbert system exists whose deduction operator coincides with H.*

As an example, in [1] one proves that all the truth functional logics of zero order whose logical connectives are interpreted by continuous functions are axiomatizable by a suitable Hilbert system. Consequently, the corresponding logical consequence operator is compact. Also, an important example is furnished by *Łukasiewicz logic with evaluated syntax*, a logic whose language is enlarged by suitable logical constants to denote the elements in $[0,1]_Q$. The semantics is the truth-functional semantics defined by the Łukasiewicz algebra. We indicate by $\mathcal{L}_{Ł}$ such a logic and by $D_{Ł}$ the related deduction operator. The following theorem is an immediate consequence of the completeness theorem for such a logic (see [23, 24, 25, 26]).

Theorem 3.5. *The logical consequence operator of Łukasiewicz logic with evaluated syntax is compact.*

4. Effectiveness

Another crucial question is the non recursive enumerability of the set of tautologies. Again, the answer of Hajek's school is to refer to the entailment relation $\models_{Varl(\otimes)}$ and therefore to the notion of general $\otimes$-tautology (see [15]). Indeed, the following important fact holds true

Theorem 4.1. *While the set of standard tautologies in Łukasiewicz first order logic is not recursively enumerable, the set of general $\otimes$-tautologies is recursively enumerable.*

So, it is sufficient to refer to the notion of general tautology to remove Pelletier's criticism.

A totally different apparatus is necessary if we will consider the *G*-approach to fuzzy logic. Indeed in this case the effectiveness has to be expressed by the "computability" of the logical consequence operator which is an operator from $[0,1]^{Form}$ to $[0,1]^{Form}$. So, we have to define such a notion in some way. This was done in [7] and [8] in the framework of effective domains theory and where also the notions of recursive enumerability (equivalently semi-decidability) and decidability for fuzzy subsets are given. Again we have to take in account the difference between the structure of $\{0,1\}^{Form}$ and $[0,1]^{Form}$. Indeed, the fact that $\{0,1\}$ is finite entails that the effectiveness in classical logic and in the *U*-approach to fuzzy logic is based on the notion of finite-steps algorithm. Instead the topological structure of [0,1] entails that in the *G*-approach the effectiveness has to be represented by endless approximation algorithms. The same kind of algorithms, for example, enabling us to claim that real-variable functions as $\sqrt{x}$, $sin(x)$, $cos(x)$,... are computable. In accordance, we propose the following definitions (see [1, 5, 7]).

Definition 4.2. Given a nonempty set *S* with a coding, we call *semi-decidable* or *recursively enumerable* a fuzzy subset $s: S\to[0,1]$ provided that a recursive map $h : S\times N \to [0,1]_Q$ exists which is order-preserving (order-reversing) with respect to the second variable and such that, for every $x\in S$,

$$s(x) = lim_{n\to\infty}h(x,n). \tag{4.1}$$

We say that *s* is *decidable* provided that both *s* and its complement are decidable.

Equivalently, we say that a fuzzy subset *s* is decidable provided that for any $x\in S$, $s(x)$ is the limit of a nested effectively computable sequence of intervals with rational bounds. There are several reasons in support of these definitions. The first one is that they are in accordance with the usual ones for subsets of *S*. Indeed it is sufficient to observe that a subset *X* of *S* is recursively enumerable in accordance with the usual definition (see for example [28]) if and only if there is a recursive map $h : S\times N \to \{0,1\}$ which is order-preserving with respect to the second variable and such that, for every $x\in S$, $c_X(x) = lim_{n\to\infty}h(x,n)$. Obviously, c_X denotes the characteristic function of *X* and the limit is defined with respect to the discrete topology in $\{0,1\}$. Another reason is that all the existing definitions of computability in fuzzy set theory are in accordance with Definition 4.2 (see for example [8] where a comparison with the existing notions of fuzzy Turing machine is done). Obviously, as in the case of famous Church thesis, it is not possible to give a definitive proof that Definition 4.2 is the best one. As an immediate consequence of (4.1) we have the following proposition.

Proposition 4.3. *Let s be semi-decidable. Then for every* $\lambda \in [0,1]_Q$ *the open* λ*-cut of s is recursively enumerable while the closed* λ*-cut belongs to the* Π_2*-level of the arithmetical hierarchy. If s assumes only a finite number of rational values, then both the open and the closed cuts of s are recursively enumerable.*

Proof. Observe that the relation $h(x,n)>\lambda$ is decidable and that the open λ-cut of *s* is

$$\{x\in S : \text{there is } n\in N \text{ such that } h(x,n)>\lambda\}$$

while the closed λ-cut is

$$\{x\in S : \text{for every } \mu\in[0,1]_Q \text{ such that } \mu<\lambda \text{ there is } n\in N,\ h(x,n)>\lambda\}.$$

The remaining part of the proposition is also evident.

The notion of semi-decidability enables us to extend the classical notion of enumeration operator (see [28]) to the operators on fuzzy subsets. We set $SEQ = F_f(S)\times S$ where $F_f(S)$ is the class of finite fuzzy subsets of *S*.

Definition 4.4. We say that a fuzzy operator $H : [0,1]^S \to [0,1]^S$ is an *enumeration operator* or a *computable operator* if a semi-decidable fuzzy subset $w : SEQ \to[0,1]$ exists such that

$$H(s)(x) = Sup\{w(s_f, x) : s_f \prec s\}. \tag{4.2}$$

This notion coincides with the one of computable operator in effective domains theory (see [7]). The following are two important properties of the enumeration operators.

Proposition 4.5. *If H is an enumeration operator then H is continuous. Moreover, for every semi-decidable fuzzy subset s, the fuzzy subset H(s) is semi-decidable.*

In [1] one proves the following theorem where a fuzzy deduction system is called *effective* provided that the fuzzy subset of logical axioms is decidable and the inference rules are computable in an uniform way.

Theorem 4.6. *The deduction operator of an effective fuzzy Hilbert system* (*in a countable language*) *is an enumeration operator. Conversely, given an enumeration operator H, an effective fuzzy Hilbert system exists whose deduction operator coincides with H.*

Observe that the converse part of this theorem is not completely satisfactory since it is obtained by admitting inference rules whose logical meaning is questionable. In the following corollary we call *complete* a theory τ such that $D_f(\tau)(\neg\alpha) + D_f(\tau)(\alpha) = 1$.

Corollary 4.7. *Given an effective fuzzy Hilbert system, if a fuzzy set of axioms τ is decidable, then the related fuzzy set $D_f(\tau)$ of consequences is semi-decidable. If τ is complete and decidable, then $D_f(\tau)$ is decidable.*

In account of the axiomatizability of Łukasiewicz first order logic with countable language, we obtain the following corollary.

Corollary 4.8. *The logical consequence operator $D_Ł$ in Łukasiewicz first order logic with countable evaluated syntax is computable. In particular, the fuzzy subset of tautologies $Tau_Ł$ is semi-decidable.*

Observe that the criticized non effectiveness of fuzzy logic is based on the fact that the (classical) set of standard tautologies, i.e. the closed 1-cut $\{\alpha \in Form : D_Ł(\emptyset)(\alpha) = 1\}$, is not recursively enumerable. In accordance with Proposition 4.3, this does not contradict the fact that the fuzzy subset $Tau_Ł$ of tautologies is semi-decidable. It means only that, given any formula α,

<u>while we are able to produce an increasing sequence of rational numbers converging to $Tau_Ł(\alpha)$, we are not able to decide if the limit of this sequence is equal to 1 or not.</u>

This phenomenon is not a characteristic of fuzzy logic since it emerges whenever a constructive approach is proposed for a notion involving real numbers. Indeed, in recursive analysis one proves the following proposition:

In the class of computable real numbers it is not decidable whether two recursive real numbers are equal or not.

Proposition 4.3 explains also why in the case we assume the set of designed values is an interval like $(\lambda,1]$, the set of tautologies is semi-decidable while in the case we assume this set is a closed interval $[\lambda,1]$, the set of tautologies is not semi-decidable (see [19]).

We conclude this section by observing that since Corollary 4.8 is a consequence of the axiomatizability of Łukasiewicz logic, it refers to a (countable) language in which there is a logical constant for every element in $[0,1]_Q$. In spite of that, this corollary holds true also if we refer to a language whose unique logical constants are 0 and 1.

Corollary 4.9. *Let $\mathcal{L}_Ł^*$ be the Łukasiewicz logic with evaluated syntax in a language with only the constants 0 and 1. Then the related logical consequence operator $D_Ł^*$ is computable. Consequently, this logic is effectively axiomatizable and the fuzzy subset of tautologies is semi-decidable.*

Proof. Taking in account the coincidence of the class of interpretations in the two logics, we have that $D_Ł^*$ is the restriction of $D_Ł$ to the fuzzy subsets of formulas in $\mathcal{L}_Ł^*$. Then $D_Ł^*$ is computable.

5. Conclusions and further criticisms

The answers to the criticisms on compactness and effectiveness in both the *U*-approach and in the *G*-approach are correct from a formal point of view, obviously. Nevertheless, we can consider these answers satisfactory only if we admit the related formalisms as adequate representations of the phenomenon we are interested in: the human everyday rational activity in which the vagueness is constantly involved. As in the case of Church Thesis, a definitive verdict on this question is not possible. Differently from Church Thesis, there are again several arguments against a claim of adequateness.

In the *U*-approach these arguments are related with the choice of the class *Varl*(⊗) to define the semantics. Indeed, in my opinion such a class is too large since it contains valuation algebras very far from human intuition. For example algebras with infinitesimal truth values. Another question is that in referring to *Varl*(⊗) we are forced to admit valuation algebras which are not complete and therefore to distinguish safe and unsafe interpretations. The completeness theorem is possible only by excluding unsafe valuations and this leads to the following question,

- *there is a way to decide in advance whether an interpretation of the predicates in a language originates a safe interpretation or not ?*

This question is of some importance, indeed, in accordance with [13], there is a formula α which is a general tautology but which is satisfied at a degree different from 1 in an unsafe model. Then in the deduction apparatus of the *U*-approach we can prove that α is completely true in spite of the fact that there is a world in which α is not completely true. This is rather disturbing.

The feeling is that in the *U*-approach the choice in favor of *VarI*(⊗) is only an instrument to obtain an axiomatizability theorem. Now, in my opinion, we have to find a syntax fitting well an early existing (natural) semantics and not a semantics fitting well a proposed syntax. On the other hand, all the students in fuzzy set theory agree in considering the semantics obtained by fixing the interval [0,1] a natural semantics. This is not surprising since the notion of graded property is based on a continuum of truth degrees and the real numbers are a good formalization of our innate idea of the continuum. Then there is something of unsatisfactory in the impossibility to refer only to the interval [0,1] and to be forced to refer to a variety of valuation structures. This remember the analogous situation in arithmetic, where the intended semantics is not axiomatizable in first order logic. So, the logicians where forced to propose a first order system of axioms which is unsatisfactory since it admits non-standard models. Obviously, the non axiomatizability of a whole logic (Łukasiewicz logic, for example) is worse than the non axiomatizability of a particular field of mathematics (arithmetic) inside of a logic (classical logic).

In any case my main criticisms are philosophical in nature and I can summarize them in the following two points.

1. <u>What's new in fuzzy logic (with respect to the tradition of multi-valued logic) is the acceptance of reasonings which are approximate as in Goguen's solution of Heap paradox</u>.

In these reasonings the available information is not necessarily a crisp set and the conclusions are not necessarily at degree 1. Moreover, due to the different topological nature of {0,1} and [0,1], it is misleading to adopt the same notion of effectiveness adopted for the inferential apparatus of classical logic. Instead,

2. <u>while effectiveness in classical logic is correctly related with the recursive arithmetic paradigm, effectiveness in fuzzy logic has to be related with the recursive analysis paradigm</u>.

Obviously, both the claims express only an opinion and no definitive proof of them is possible.

G-approach avoids some of these criticisms. Mainly, it shows that also in the case the valuation structure is fixed and the truth values are real numbers, a logic with a related completeness theorem is possible. As emphasized in this paper, this logic satisfies suitable compactness and effectiveness properties. Unfortunately, I am aware that also the *G*-approach is not free of further difficulties. Firstly, as in the *U*-approach, one requires the introduction in the language of logic constants for the rational number in [0,1]. This is an ingenious trick but it is not representative of the human logical activity since none consider ¾ as a sentence (the result of Corollary 4.9 is not a convincing answer since the proposed inference rules are not natural at all). Moreover, the *G*-approach is not sufficiently flexible. For example, in assigning a fuzzy subset of hypotheses we have to put a precise lower-bound constraint to the truth value of a formula while it should more natural to assign a fuzzy constraint. A lack of flexibility is also apparent in the proposed notion of fuzzy model of a fuzzy theory which is based on Zadeh's crisp inclusion. Indeed, let m be a fuzzy model of a fuzzy theory τ. It is evident that both the assignments m and τ cannot be considered definitive and precise since m depends on the subjective modeling of the vague predicates and τ depends on the subjective valuation of the truth degree of the formulas. Now assume that either m or τ is subject to a slightly variation as a consequence of a tuning process, an essential component in all the applications in fuzzy mathematics. Then it is possible that m ceases completely to be a model of τ while it should be natural to expect m is again a model of τ at some degree. This suggests that it should be opportune to reformulate the notion of fuzzy model of a fuzzy theory by substituting the crisp inclusion with a graded inclusion. More in general, perhaps vagueness entails a semantics in which, differently from Tarski's paradigm, notions as

"distance between models", "flexibility", "learning", "tuning", "evolutionary meaning", "linguistic game", "negotiation"

play a crucial role. On the other hand, it is not only for technical reasons that these notions are present in all the applications of fuzzy logic and, in particular, in fuzzy control.

Concluding, my persuasion is that we are far from a definitive answer to the question whether is possible to formalize those human inferential processes in which vague notions are involved. In spite of that, I am trusting that the main difficulties will be resolved by redefining in a more flexible way the basic formalisms. We cannot even exclude that the research for a general theory is an impossible task and that we have to be satisfied in fragments of fuzzy logic able to formalize processes in which, for example, the involved pieces of information are not too complicate. Fuzzy logic programming and (the logical approaches to) fuzzy control are examples in such a direction (see [6] and [31]).

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